

## Theory and Methodology

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# On the stability region for multi-level inventory problems \*

Knut RICHTER

*Technical University, Faculty of Economics, Karl-Marx-Stadt, German Democratic Republic*

József VÖRÖS

*Janus Pannonius University, Faculty of Economics, Pécs, Rákoczi u. 80, H-7622, Hungary*

**Abstract:** This paper focuses on analysing the multi-stage assembly system with cost function, which is widely used in the literature. We shall point out that the set of cost inputs having the same optimal production plan is a convex cone. In addition, the structure of an optimal solution is analysed to reveal the stability region.

**Keywords:** Production planning

### 1. Introduction

The dynamic lot-size model is one of the best known standard models in OR/MS, and the procedures for solving the problem have received considerable attention in the literature. In addition to the optimal dynamic programming and branch-and-bound algorithms, numerous heuristics have also been developed for both single- and multi-level problems. However, relatively little effort has been made to investigate the stability of a schedule. The stability region of a schedule means the set of cost inputs having the same production plan for a given demand series.

This question is of interest for both theory and practice. It would be useful for the practitioner to know the range of cost parameters over which the

optimal production is not altered. Characterizing the shape of the stability region is the theoretical question of interest.

The single-level lot-sizing stability problem was analysed by Richter (1987). Using constant set-up and holding costs and the assumption that the cost inputs belonging to the stability region have the same production plan for every problem with period  $t$ ,  $t = 1, 2, \dots, T$ , where  $T$  is the length of the planning horizon, he gave the explicit form of the stability region. He also pointed out that the stability region is a convex cone. Omitting the need of these strong assumptions, we show that this convex cone property can be extended to more general multi-level problems with certain cost functions. Analysing the structure of an optimal schedule, we also show that this production plan can be expressed by a regeneration matrix. The advantage of this production-quantity-independent plan definition is that this could open the way to a discussion on the impact of the changes in demand.

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The paper is organized as follows. Section 2 states the multi-level problem and shows that the stability region is a convex cone. In Section 3 we analyse the structure of an optimal solution and in Section 4 the main results are summarised and ideas for further research are provided.

**2. The stability region of the multi-stage assembly system**

In a multi-stage assembly system, manufacturing of an item requires a certain number of components and, in turn, is itself a component of a single parent item. The product structure in this production process can be represented by a directed acyclic network where the set of nodes represents the set of items and the set of directed arcs denotes the processing operations. Let the number of items be  $M$  and let  $M$  be the only facility which produces assemblies used to supply the customer's demand. Raw materials are available in unlimited amounts as input to source facilities. All facilities are allowed to carry inventories; facilities 1 to  $M - 1$  carry in-process inventories while the  $M$ th facility carries the finished good inventory. It is assumed that production and shipment are instantaneous, and that one unit of production on facility  $m$  requires one unit of input from every facility  $k$ ,  $k \in A(m)$ , where  $A(m)$  is the set of immediate predecessors of  $m$ . Backlogging of demand is not allowed. (It is worth noticing that more general product structures can be transformed to this assembly system—see Afentakis and Gavish (1986).)

Let  $B(m)$  denote the unique immediate successor, while  $P(m)$  and  $R(m)$  denote the set of all predecessors and successors of node  $m$ , respectively. Then, in the assembly system represented by Figure 1 we have  $B(2) = \{4\}$ ,  $A(4) = \{2, 3\}$ ,  $R(1) = \{2, 4\}$ , and  $P(4) = \{1, 2, 3\}$ .

Let  $d_t$  denote the demand in period  $t$ ; it is assumed that the demand is known for periods 1 to  $T$ . Let  $X_{mt}$  denote the production at facility  $m$

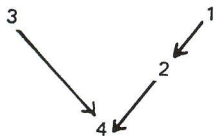


Figure 1. An assembly system

in period  $t$ ; the cost of this production is denoted  $C_{mt}(X_{mt})$ .  $I_{mt}$  is the inventory at the end of period  $t$  at facility  $m$  and the corresponding holding cost is  $H_{mt}(I_{mt})$  for all  $m \in \langle 1, M \rangle$  and for all  $t \in \langle 1, T \rangle$ , where

$$\langle a, b \rangle = \{ a, a + 1, \dots, b \}.$$

Let  $B(M) = \{ M + 1 \}$ .

Then the multi-stage assembly problem, problem (1), can be written as

$$\text{Minimize } \left\{ \sum_{m=1}^M \sum_{t=1}^T (C_{mt}(X_{mt}) + H_{mt}(I_{mt})) \right\} \tag{1a}$$

subject to

$$I_{mt} = I_{m,t-1} + X_{mt} - X_{B(m),t} \tag{1b}$$

$$X_{M+1,t} = d_t \text{ for all } t \in \langle 1, T \rangle, \tag{1c}$$

$$I_{m0} = I_{mT} = 0, \quad I_{mt} \geq 0, \quad X_{mt} \geq 0 \tag{1d}$$

When cost functions  $C$  and  $H$  are concave, Veinott (1969) showed that a node can have at most one positive input in an extreme point solution, i.e.,  $I_{m,t-1} X_{mt} = 0$ . Now let

$$C_{mt}(X_{mt}) = \begin{cases} s_m + c_m X_{mt} & \text{for } X_{mt} > 0, \\ 0 & \text{for } X_{mt} = 0, \end{cases}$$

and

$$H_{mt}(I_{mt}) = h_m I_{mt}, \quad h_{B(m)} \geq h_m,$$

for all  $m \in \langle 1, M \rangle$  and  $t \in \langle 1, T \rangle$ ;

then (1a) can be written as

$$\text{Minimize } \left\{ \sum_{m=1}^M \left( s_m \sum_{t=1}^T y_{mt} + c_m \sum_{t=1}^T X_{mt} + h_m \sum_{t=1}^T I_{mt} \right) \right\},$$

where  $y_{mt} \in \{0, 1\}$ ,  $X_{mt} \leq N y_{mt}$ , and  $N$  is a large positive number, or

$$\text{Minimize } \left\{ \sum_{m=1}^M \left( s_m \sum_{t=1}^T y_{mt} + d c_m + h_m \sum_{t=1}^T I_{mt} \right) \right\}, \tag{1a}'$$

where  $d = \sum_{t=1}^T d_t$ .

Further, let there be a demand series given. For cost inputs we use the following notations:  $s' = (s_1, \dots, s_M)$ ,  $h' = (h_1, \dots, h_M)$ ,  $c' = (c_1, \dots, c_M)$ , and  $d' = (d, d, \dots, d)$ , where primes denote transposition. Now let us consider an optimal solution  $X_{m_t}^o$  of problem (1) with cost function (1a)' and let matrix  $X^o$  be of order  $M \times T$  with  $X^o = [X_{m_t}^o]$ .

Let  $SR(X^o)$  denote the set of cost inputs for which problem (1) has an optimal schedule with production plan  $X^o$ .  $SR(X^o)$  is called the stability region.

**Theorem 1.**  $SR(X^o)$  is a convex cone in  $R^{3M}$ .

**Proof.** If  $(s'c'h') \in SR(X^o)$ , then it can easily be seen that  $\lambda(s'c'h') \in SR(X^o)$ ,  $\lambda \geq 0$ , by multiplying the entire cost function by a constant. Let  $(\hat{s}'\hat{c}'\hat{h}')$  be a cost vector for which there is no  $\lambda$  such that

$$\lambda(s'c'h') = (\hat{s}'\hat{c}'\hat{h}').$$

If  $(\hat{s}'\hat{c}'\hat{h}') \in SR(X^o)$ , then we have to show that  $(\tilde{s}'\tilde{c}'\tilde{h}') \in SR(X^o)$ , where

$$(\tilde{s}'\tilde{c}'\tilde{h}') = \lambda(s'c'h') + (1 - \lambda)(\hat{s}'\hat{c}'\hat{h}')$$

and  $0 \leq \lambda \leq 1$ . But if a solution is optimal for two cost vectors, then it is optimal for their sum, too.

□

(It is easy to see that the theorem is also true for time-dependent costs.)

### 3. Analysing the structure of an optimal solution

Next we show that  $X^o$  can be defined by a regeneration matrix. For this purpose let us analyse the structure of an optimal solution of problem (1).

**Definition.** A schedule is *nested* if  $X_{m_t} > 0$  implies that  $X_{B(m),t} > 0$ . Period  $t$  is an  $m$ -level regeneration point in a  $T$  period problem if  $I_{jt} = 0$  for all  $j \in R(m)$ .

Crowston and Wagner (1973) pointed out that, with cost function (1a)', problem (1) has a nested optimal schedule.

Let  $l_m(t)$  and  $t$  be adjacent  $m$ -level regeneration points in a  $T$  period problem,  $l_m(t) < t$ , and matrix  $J_T$  be of order  $M \times T$ ;  $J_T = [j_{m,k}]$ , where

$$j_{m,k} = \begin{cases} l_m(k) & \text{if } k \text{ is an } m\text{-level regeneration} \\ & \text{point in a } T \text{ period problem,} \\ k & \text{otherwise.} \end{cases}$$

$J_T$  is called the  $T$  period regeneration matrix.

**Production Condition.**  $X_{m_t} > 0$  iff period  $t - 1$  is an  $m$ -level regeneration point.

**Lemma 1.** A schedule satisfying the Production Condition is a nested schedule.

**Lemma 2.** Let a regeneration matrix  $J_T$  be given. Then the following schedule defined on  $J_T$  is nested:

$$X_{m,l_m(t)+1} = \begin{cases} X_{B(m),t} & \text{if } l_m(t) + 1 = t, \\ \sum_{r=l_m(t)+1}^t X_{B(m),r} & \text{if } l_m(t) + 1 < t, \end{cases} \quad (2)$$

and

$$X_{m,r} = 0 \quad \text{if } j_{m,r-1} = r - 1 \\ \text{for } m \in \langle 1, M \rangle \quad (X_{M+1,t} = d_t).$$

**Proof.** The schedule satisfies the Production Condition thus it is nested. □

Let  $s^m(t)$  and  $h^m(t)$  denote the total number of set-ups and the total inventory, respectively, at the  $m$ -th stage in the first  $t$  periods with the provision that period  $t$  is an  $m$ -level regeneration point. Let

$$h_{l,t}^m = \sum_{r=l+1}^t (r - l - 1) X_{B(m),r} \quad \text{for } l < t, \quad (3)$$

with the assumption that  $l$  and  $t$  are neighbouring  $m$ -level regeneration points.

Thus we can write

$$h^m(0) = 0, \\ h^m(t) = h^m(l_m(t)) + h_{l_m(t),t}^m, \quad (4)$$

and

$$s^m(0) = 0, \\ s^m(t) = s^m(l_m(t)) + 1, \quad (5)$$

where  $t$  is an  $m$ -level regeneration point in the  $T$ -period problem.

**Example.** Let

$$J_3 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix},$$

i.e.,  $M = 2, T = 3$  and  $A(2) = \{1\}$ . Then,

$$\begin{aligned} s^1(3) &= 2, & h^1(3) &= 0, \\ s^2(3) &= 2, & h^2(3) &= d_2. \end{aligned}$$

The schedule defined by  $J_3$  is shown in Figure 2.

**Lemma 3.** Let  $J_T$  be a regeneration matrix and  $C_T(s'c'h')$  be the cost value of an optimal nested schedule for cost inputs  $(s'c'h')$  in the  $T$  period problem. Then,

$$C_T(s'c'h') = s's(T) + c'd + h'h(T),$$

where

$$\begin{aligned} s(T)' &= (s^1(T), \dots, s^M(T)), \\ h(T)' &= (h^1(T), \dots, h^M(T)). \end{aligned}$$

The proof of Lemma 3 simply follows from the definition of the parameters.

**Lemma 4.**  $(s'c'h') \in SR(J_T)$  iff there is no  $\hat{J}_T$  for which

$$s'\hat{s}(T) + c'd + h'\hat{h}(T) < s's(T) + c'd + h'h(T), \tag{6}$$

where  $\hat{s}(T), \hat{h}(T)$  and  $s(T), h(T)$  are defined on  $\hat{J}_T$  and  $J_T$ , respectively.

As, for example with the help of  $T$ -element binary numbers beginning with 1, all regeneration matrices can easily be generated, Lemma 4 offers

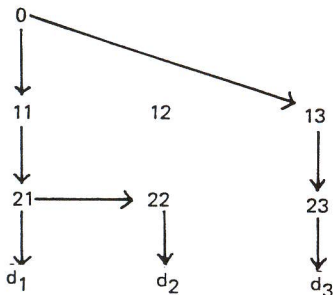


Figure 2. The schedule defined by  $J_3$

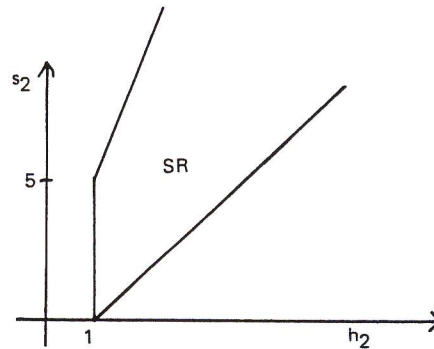


Figure 3. The stability region of facility 2, provided that  $s_1 = 5, h_1 = 1$

a simple enumeration procedure for exhibiting the stability region. Then, applying (6) for  $J_3$  defined earlier,  $SR(J_3)$  is determined by the following inequality system for  $d_1 = 1, d_2 = 2, d_3 = 5$ :

$$\begin{aligned} 3s_1 + 3s_2 &\geq 2s_1 + 2s_2 + 2h_2, \\ 2s_1 + 3s_2 + 5h_1 &\geq 2s_1 + 2s_2 + 2h_2, \\ 2s_1 + 3s_2 + 2h_1 &\geq 2s_1 + 2s_2 + 2h_2, \\ &\vdots \\ s_1 + s_2 + 12h_2 &\geq 2s_1 + 2s_2 + 2h_2. \end{aligned}$$

Fixing  $s_1 = 5$  and  $h_1 = 1$  the schedule given by  $J_3$  does not change if  $s_2$  and  $h_2$  satisfy the following system:

$$10h_2 - 5 \geq s_2 \geq 2h_2 - 2, \quad s_2 \geq 0, \quad h_2 \geq 1.$$

Figure 3 shows the stability region of facility 2.

#### 4. Summary

This paper has analysed the multi-stage assembly system and stated that its stability region is a convex cone. It is shown that an optimal nested schedule can be expressed with the help of a regeneration matrix which is useful in the enumeration procedure used for revealing the stability region. The regeneration matrix is a quantity-independent plan representation. However, the enumeration procedure is exponential in complexity. For the facilities in the series inventory problem, a dynamic programming type (see Chand, 1983) procedure would increase the quickness of computation if we were interested in revealing the stability region of costs having the same  $m + 1$  level

regeneration points for every possible combination of neighbouring  $m$  level regeneration points,  $m \in \langle 1, M - 1 \rangle$ . (This would mean the direct extension of Richter's result for the multi-stage problem using the solution of the linear inequality system.) However, even for the single-level problem, practice does not need this assumption. Thus, the challenging task for the researchers in this field remains: to find, under mild conditions, effective algorithms for revealing  $SR(J_T)$ .

Because of its practical importance, further research may tend to analyse the effect of changes in demand forecasting (for single-level problems, see Richter and Vörös, 1988).

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