An EOQ repair and waste disposal model

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Revised version

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Summary:

A two stage EOQ model for manufacturing, repairing and disposing products is discussed. First the lot size function and the minimal cost function is derived for the waste disposal rate of products and an economically optimal waste disposal rate is determined. Secondly this rate is regarded as a function of the waste disposal cost (or price) and the behavior of this convex-concave function is analysed. Third, in a dual problem the waste disposal price is analysed which maximizes the income of a price dictator.

Keys: EOQ model, waste disposal, cost minimization, income maximization

1. The model

1.1. Introduction

A two stage EOQ model describing the manufacturing (or order) of new and the repair of used products (for instance, containers) in a first shop and the employment of the products in a second shop is offered in this paper. The used products can either be stored at the second shop and after some time be brought back to the first shop for repair, or be disposed somewhere outside. For the first shop economic order quantities have to be determined for new products and for repairable products in order to meet the constant demand rate of the second shop. Some of the used products are collected at the second shop according to a certain not necessarily unique repair rate. The share of the products not provided for repair is called waste disposal rate.

Ordinary multi stage EOQ models have been studied long ago [2,5,6]. Recently a paper appeared which also focuses on repairing and scrapping products (comp. [4]). For the last years EOQ models have been used extensively to explain several production phenomena. Some overview showing the various applications is provided in [1]. Our paper can be regarded as another attempt to apply the EOQ framework to production situations, this time to problems appearing with the alternatives to repair or to dispose used products.

The repair rate, which displays the ecological behavior of the producer, is fixed in [4]. Pushing this rate up one might contribute to an increasing ecological orientation of the production, and it is clearly of interest to trace the economical consequences of alternative repair and waste disposal rates to production. As in other studies, the simplicity of the EOQ model allows to show directly the relationship between model inputs, here between the ecological cost (waste disposal price) and ecological behavior (waste disposal rate). If more practical situations are modelled, this chance of expressing the explicit relationship between ecological and economical parameters probably gets lost. Other studies are, of course, needed to prove the applicability of the presented approach in a more practical framework.

1.2. Assumptions

The model is based on the following assumptions:

A Technological assumptions:

(i) A first shop is providing a homogenuous product used by a second shop at a constant demand rate of d units per time unit.

(ii) The first shop is manufaturing new products and also repairing products used by the second shop. The repaired products are then regarded as new.

(iii) The products are used by the second shop and collected there according to a repair rate β. The other products are immediately disposed as waste outside (waste disposal rate $\alpha = 1 - \beta$).

(iv) After some variable time interval T the collected products are brought back to the first shop and will be repaired. If the repaired products are finished the manufacturing process starts to cover the remaining demand for the time interval.

(v) The processes of manufacturing, repairing and using the products are instantaneous.

The inventory stocks occuring in this system are illustrated by Fig. 1

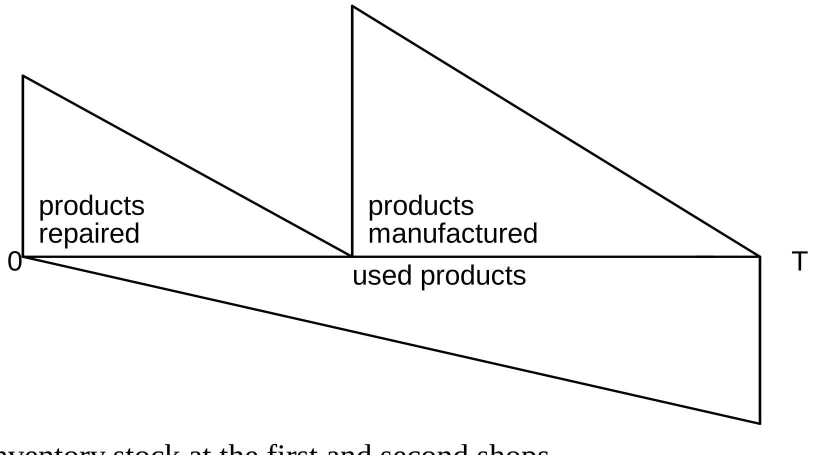

Fig. 1: Inventory stock at the first and second shops

B Cost assumptions:

(i) Fixed cost for a time interval: M

(ii) Per unit cost/price of manufacturing, repairing and disposing products: b, k, e

(iii) Per unit per time unit holding cost at first and second shop: h, u

C Notations:

(i) Waste disposal rate and repair rate: $\alpha, \beta, \alpha+\beta=1$

(ii) Demand rate: d

(iii) Lot size of a time interval T: x

(iv) Over all cost for a time interval: K_Z

(v) Size of disposal and repair lots: $\alpha x, \beta x$

D Objectives:

(i) The minimal per time unit over all cost for the producer is to be determined.

(ii) The maximal per unit income of the disposal price dictator is to be determined.

1.3. The model and major properties

The over all cost for a variable time interval $[0, T]$ is

$$K_Z = M + \alpha Td(b+e) + h\alpha_{\llcorner} x_{\llcorner}/2d + \beta Tdk + h\beta_{\llcorner} x_{\llcorner}/2d + u\beta Tx/2. \qquad (1)$$

According to the first objective, the per time unit cost for the producer is to be minimized:

$$K = K_Z/T = dM/x + \frac{x}{2}[(\alpha^2 + \beta^2)h + \beta u] + d[\alpha(b+e)+\beta k] \to \min,$$

or

$$K = dM/x + \frac{x}{2} H(\alpha) + d\, P(\alpha), \qquad (2)$$

where the inventory cost

$$H(\alpha) = (\alpha^2 + \beta^2)h + \beta u = 2\alpha^2 h - \alpha(2h+u)+h+u \qquad (3)$$

and the production cost

$$P(\alpha) = \alpha(b+e)+\beta k = k - \alpha(k-b-e)$$

are regarded as functions of α (and of β, if this is of greater interest.)

The optimal lot size is

$$x^* = \sqrt{\frac{2dM}{h(\alpha^2+\beta^2)+u\beta}} = \sqrt{\frac{2dM}{H(\alpha)}}. \tag{4}$$

and the lots of disposed and repaired products are given by αx^* and βx^*.

The minimal cost function is then

$$K^* = \sqrt{2dM(h(\alpha^2+\beta^2)+u\beta)} + d((b+e)\alpha + k\beta) = \sqrt{2dMH(\alpha)} + dP(\alpha). \tag{5}$$

It can be shown that for arbitrary u, $h > 0$ and for all $\alpha \in [0,1]$ the inequation $H(\alpha) > 0$ holds, so that the square root terms above are always real.

2. Analysis of the minimal cost function

2.1. The optimal waste disposal rate

Let the waste disposal rate be variable within some range $[\alpha_{min},\alpha_{max}]$. The question is which rate is minimal with respect to function (5). First conditions for the convexity of the function (5) are provided:

Lemma 1: The function (5) is convex in α iff

$$4h(h+u) > u^2 \tag{6}$$

holds.

Proof: If the formula (3) is used to express the cost function K, then

$$K^{*\prime} = H'(\alpha) \cdot \sqrt{2dM} / 2\sqrt{H(\alpha)} + (e-k)$$ holds.

The second derivative is then $K^{*''} = +\sqrt{dM/4} \cdot \dfrac{H''(\alpha)H(\alpha) - H'^2(\alpha)/2}{H(\alpha)^{3/2}}$.

A straightforward analysis shows that for the second derivative

$$K^{*''} = C \cdot [4h(h+u) - (2h+u)^2/2] = C \cdot (2h(h+u) - u^2/2)$$

holds, with a positive constant C. Then $K^{*''} > 0$ if and only if the condition of the theorem holds.

The set of points (h, u) which fulfil condition (6) forms a convex cone in R^2_+.
It can be shown that the lot size function $x^*(\alpha)$ is concave under the same conditions.

Example: Let $M = \$ 100$, $d = 10$ *units*, $h = \$ 6$, $u = \$ 4$, $k = \$ 6$, $b = \$ 0$, $e = \$ 8$
$\alpha \in [0.1, 0.9]$.

Then the function (4) is illustrated by Fig. 2.

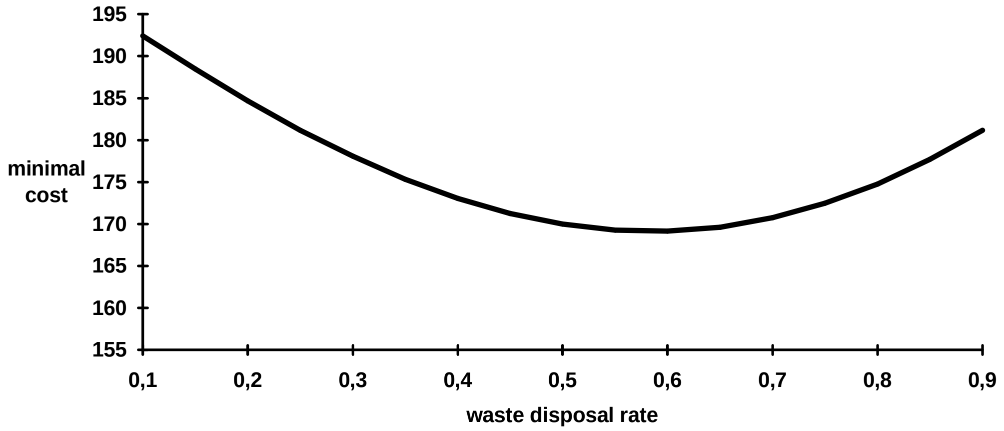

Fig. 2: Minimal cost as a function of the waste disposal rate

If $\alpha = 0.5$ then lot size is $x^* = 20$ and the minimal cost is $K^* = \$ 170$. Since this function is convex, the minimum can be found again by the application of calculus. If the value of the disposal rate is

$$\alpha^* \in [\alpha_{min}, \alpha_{max}],$$

then it is the optimal rate. In the other case one of the bounds is optimal. For the given example the optimal disposal rate is $\alpha^* = 0.5855$. *

2.2. The optimal waste disposal rate as a function of the waste disposal price

Now, the value α^* is studied to see how it reacts at changes in the waste disposal price *e.*

Theorem: Provided the inequality (6) holds, the optimal waste disposal rate is

$$\alpha^*(e) = \frac{2h+u}{4h} + \frac{k-b-e}{4h}\sqrt{\frac{d(4(h+u)h-u^2)}{4hM-d(k-b-e)^2}}, \qquad (7)$$

if this value is feasible. In the other case one of the bounds is the optimal waste disposal rate.

Proof: Let $\delta = k - b - e$ and $H(\alpha) = a\alpha^2+b\alpha+c$, $H'(\alpha) = 2a\alpha+b$, $H''(x) = 2a$
with $a = 2h$, $b = -2h-u$, $c = h+u$.
Then

$$K^{*\prime} = \sqrt{\frac{dM}{2(a\alpha^2+b\alpha+c)}}(2a\alpha+b) - d\delta = 0,$$

i. e. $\sqrt{dM/2}\cdot(2a\alpha+b) = d\delta\cdot\sqrt{a\alpha^2+\alpha\alpha+c}$ is to be solved.

That leads to

$$\alpha^2 + b/a\cdot\alpha + \frac{Mb^2-2d\delta^2c}{4a^2M-2ad\delta^2} = 0,$$

with the solutions

$$\alpha = -b/2a \pm \sqrt{\frac{b^2}{4a^2}-\frac{2d\delta^2c}{2ad\delta^2}} = -b/2a \pm \frac{|\delta|}{2a}\sqrt{\frac{d(4ac-b^2)}{2aM-d\delta^2}}.$$

Replacing the parameters *a, b, c* and discussing the signs formula (7) arises.
The cost function is convex due to the assumption (6) and the waste disposal rate is optimal because of this convexity. ¨

For the example the optimal waste disposal rate is illustrated in Fig. 3.

Fig. 3: Optimal waste disposal rate as a function of the waste disposal price

The expression (7) and the figure make clear that the optimal waste disposal rate is a convex-concave function of the waste disposal price. Then price changes influence the disposal rate quite differently: For small and for large price inputs the rate is more sensitive to the price variation than in the case of moderate price inputs. If the cost inputs are very small or very large, then there is no sensitivity at all, because the disposal rate forced by *e* lies outside the feasibility region.

The impact of *e* on α^* can also be expressed in a different way by rewriting (7) in terms of inventory costs:

$$\alpha^*(e) = \alpha^\circ + \delta \sqrt{\frac{H(\alpha^\circ)}{2h(\frac{4hM}{d} - \delta^2)}}\ , \qquad (8)$$

where α° is the disposal rate which minimizes the inventory cost function $H(\alpha)$. Formula (8) shows that α^* depends on *e* only by the cost difference $\delta = k - b - e$. If $\delta = 0$ (cost balance) then $\alpha^* = \alpha^\circ$ minimizes $H(\alpha)$ because in this case only the total holding cost is relevant. In the other case α° is shifted by a term depending on δ.

If the holding cost is the same for both the shops, i. e. $u = h$, then for the cost balance case holds $\alpha^*(k-b) = 3/4$. In other words, in this case it is optimal to repair 25% and to dispose 75% of the product. The convex-concave behavior of the optimal waste disposal rate is proved below.

Lemma 2: Provided relation (6) holds, $\alpha^*(e)$ is a convex-concave function.

Proof: First, let the case $e < k\text{-}b$ be considered. Then the variable part $g(e)$ of the formula (8) will be studied. Denoting the constants by C and D the formula

$$g(e) = \sqrt{\frac{C(k-b-e)^2}{D-(k-b-e)^2}}$$

arises. The straightforward analysis shows this function to be convex. Then, however, $\alpha^*(e)$ is also convex. The concavity of the function for the other case $e > k\text{-}b$ follows by analoguous reasons. ¨

3. The optimal waste disposal price

3.1. The problem

Let the waste disposal price e appear outside, for instance be dictated by some authority. If, according to the second objective, this authority is maximizing its per time unit income from the disposal activities of the cost minimizing producer, it tries to maximize the function

$$f(e) = d\alpha^*(e)e, \tag{9}$$

subject to some restriction for the price $e \in [e_{min}, e_{max}]$, provided it has sufficient information about the producer. Since the constant d can be dropped, only $f(e) = \alpha^*(e)e$ will be studied. Furthermore only the case of

$$\alpha^*(e_{min}) < \alpha_{max} \text{ and } \alpha^*(e_{max}) > \alpha_{min} \tag{10}$$

is considered here. These inequalities secure that the maximal income will not be provided by prices forcing the producer to apply the minimal or maximal waste disposal rates.

For the above example this function is shown in Fig. 4.

Fig. 4: Income as a function of the waste disposal price for the feasible price region [1, 16]; the behaviour of *f(e)* outside this region is indicated by a dotted line.

It can be seen that there is such a waste disposal price which maximizes the income. Unfortunately, no analytic expression for the income maximizing disposal price could be found so far. Fig. 4 and simulation makes clear that the optimal price is somewhere in the neighborhood of $e^* = \$ 10.50$ with $f(e^*) = \$ 5.0123$ and $\alpha^*(e^*) = 0.48$.

The price dictator gains a per time unit income of $d \cdot f(e^*) = \$ 50.123$, if it sets a price of *$ 10.50* and forces by this price the cost minimizing producer to dispose *48%* of its products as waste.

3.2. The nearly optimal waste disposal price

The optimal price can be determined roughly if the disposal rate is replaced by its linear approximation

$$g(e) = \alpha^\circ + \rho(e+b-k) \qquad (10)$$

with $\rho = \dfrac{\alpha_{\max} - \alpha_{\min}}{e_1 - e_2}$, $\alpha^\circ = \alpha^*(k-b)$, $\alpha^*(e_1) = \alpha_{\max}$, and $\alpha^*(e_2) = \alpha_{\min}$.

Then the problem (8) reduces to $f^*(e) = g(e)e = (\alpha^\circ + \rho(b-k))e + \rho e^2 \rightarrow \max$, (11)

and calculus will provide the "optimal" price: $e' = \dfrac{\alpha^\circ + \rho(b-k)}{2\rho}$. (12)

For the data used so far, i. e. for *M= \$ 100, d=10, k= \$ 6, h= \$ 6, u= \$ 4, b= \$ 0,* $\alpha_{\min} = 0.1$, $\alpha_{\max} = 0.9$, the value $\rho \approx -0.8/16 = -0.05$ can be used as approximation. Then $e' = \$ (20/3 + 3) = \$ 9.66$ holds, i. e. the disposal price is near the optimal one.

4. Conclusion

In this very simple situation an optimal waste disposal rate exists and the reaction of a cost minimizer to the changes of the waste disposal price can be traced. It will be of interest to ask the same questions for problems with many items, for a convex repair cost function and for holding cost depending on the repair rate. An analytic expression for the optimal price would give the chance to trace also the price reaction on the changes of other model inputs.

References:

[1] Groenevelt, H.; Pintelon, L.; Seidmann, A.; Production lot sizes with machine break-downs, Management Science 38(1992)1, 104-123

[2] Hahm, J.; Yano, C.A.; The economic lot and delivery scheduling problem: The single item case, Int. Journal of Production Economics, 28(1992) 235-252

[3] Müller-Merbach, H.; Optimale Losgrößen bei mehrstufiger Fertigung, Zeitschrift wirtschaftliche Fertigung 60(1965) 113-118

[4] Mabini M. C.; Pintelon, L. M.; Gelders, L.F.; EOQ type formulations for controlling repairable inventories, Int. Journal of Production Economics, 28(1992), 21-33

[5] Stadtler, H.; Optimale Lospolitiken für eine Leichtmetallgießerei, OR-Spektrum, 14(1992) 217-227

[6] Taha, H.A.; Skeith, R.W.; The economic lot sizes in multistage production systems, AIIE Transactions 2(1970) 157-162

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